

Arguments Containing Compound Statements

2.1 Simple and Compound Statements

All statements can be divided into two kinds, simple and compound. A *simple* statement is one that does not contain any other statement as a component part, whereas every *compound* statement does contain another statement as a component part. For example, 'Atmospheric testing of nuclear weapons will be discontinued or this planet will become uninhabitable' is a compound statement that contains, as its components, the two simple statements 'Atmospheric testing of nuclear weapons will be discontinued' and 'this planet will become uninhabitable'. The component parts of a compound statement may themselves be compound, of course.

As the term 'component' is used in logic, not every statement that is part of a larger statement is a component of the larger statement. For example, the last six words of the statement, 'The third wife of Bertrand Russell was a beautiful girl', can be regarded as a statement in its own right. But it is not a *component* part or a *component* of the larger statement containing it. For a part of a statement to be a component of a larger statement, two conditions must be satisfied. First, the part must be a statement in its own right; and second, if the part is replaced in the larger statement by any other statement, the result of that replacement must be meaningful. Although the first condition is satisfied in the example given, the second is not. For if the part 'Bertrand Russell was a beautiful girl' is replaced by the statement 'Where there's smoke, there's fire', the result of that replacement is arrant nonsense.¹

The statement 'Roses are red and violets are blue' is a *conjunction*, a compound statement formed by inserting the word 'and' between two statements. Two statements so combined are called *conjuncts*. The word 'and' has other uses, however, as in the statement 'Castor and Pollux were twins', which is not compound, but a simple statement asserting a relationship. We introduce the dot '.' as a special symbol for combining statements conjunctively. Using

this notation, the preceding conjunction is written 'Roses are red · violets are blue'. Where p and q are any two statements whatever, their conjunction is written $p \cdot q$.

Every statement is either true or false, so we can speak of the *truth value* of a statement, where the truth value of a true statement is *true* and the truth value of a false statement is *false*. There are two broad categories into which compound statements can be divided, according to whether or not there is anything other than the truth values of its component statements that determines the truth value of the compound statement. The truth value of the conjunction of two statements is completely determined by the truth value of its conjuncts. A conjunction is true if both its conjuncts are true, but false otherwise. For this reason a conjunction is a *truth-functional* compound statement, and its conjuncts are *truth-functional* components of it.

Not every compound statement is truth-functional. For example, the truth-value of the compound statement 'Percival Lowell believed that Mars is inhabited' is not in any way determined by the truth value of its component simple statement 'Mars is inhabited'. This is so because it could be true that Percival Lowell believed that Mars is inhabited, regardless of whether it is inhabited or not. A person can believe one truth without believing another, and a person can believe one falsehood without believing all of them. So the component 'Mars is inhabited' is not a truth-functional component of the compound statement 'Percival Lowell believed that Mars is inhabited,' and the latter statement is not a truth-functional compound statement.

We define an occurrence of a component of a compound statement to be a *truth-functional component* of that compound statement provided that: If that occurrence of the component is replaced in the compound (or in any component of the compound statement which contains the component in question) by different statements that have the same truth value as each other, the different compound statements produced by these replacements will also have the same truth values as each other. And now a compound statement is defined to be a *truth-functional compound statement*, if all occurrences in it of its components are truth-functional components of the statement.²

The only compound statements we shall consider here will be truth-functionally compound statements. Therefore, in the rest of this book we shall use the term "simple statement" to refer to any statement that is not truth-functionally compound.

Since conjunctions are truth-functionally compound statements, our dot symbol is a truth-functional connective. Given any two statements p and q there are just four possible sets of truth values they can have, and in every case the truth value of their conjunction $p \cdot q$ is uniquely determined. The four possible cases can be exhibited as follows:

¹This account of compound and component statements has been suggested by Professor C. Mason Myers of Northern Illinois University, Professor Alex Blum of Bar-Ilan University, and Professor James A. Martin of the University of Wyoming. See Martin's 'How Not to Define Truth-Functionality,' *Logique et Analyse*, 1970, 14^e Année, N° 52, pp. 476-482.

²Compare this definition with the one proposed by Professor David H. Sanford in his 'What Is a Truth Functional Component?' *Logique et Analyse*, 1970, 14^e Année, N° 52, pp. 483-486.

in case p is true and q is true, $p \cdot q$ is true;
 in case p is true and q is false, $p \cdot q$ is false;
 in case p is false and q is true, $p \cdot q$ is false;
 in case p is false and q is false, $p \cdot q$ is false.

Representing the truth values true and false by the capital letters 'T' and 'F', respectively, the way in which the truth value of a conjunction is determined by the truth values of its conjuncts can be displayed more briefly by means of a *truth table* as follows:

p	q	$p \cdot q$
T	T	T
T	F	F
F	T	F
F	F	F

Since it specifies the truth value of $p \cdot q$ in every possible case, this truth table can be taken as *defining* the dot symbol. Other English words such as 'moreover', 'furthermore', 'but', 'yet', 'still', 'however', 'also', 'nevertheless', 'although', and so forth, and even the comma and the semicolon, are also used to conjoin two statements into a single compound one, and all of them can be indifferently translated into the dot symbol, so far as truth values are concerned.

The statement 'It is not the case that lead is heavier than gold' is also compound, being the *negation* (or *denial* or *contradictory*) of its single component statement 'lead is heavier than gold'. We introduce the symbol ' $\sim$ ', called a *curl* (or a *tilde*) to symbolize negation. There are often alternative formulations in English of the negation of a given statement. Thus where L symbolizes the statement 'lead is heavier than gold', the different statements 'it is not the case that lead is heavier than gold', 'it is false that lead is heavier than gold', 'it is not true that lead is heavier than gold', 'lead is not heavier than gold' are all indifferently symbolized as $\sim L$. More generally, where p is any statement whatever, its negation is written $\sim p$. Since the negation of a true statement is false and the negation of a false statement is true, we can take the following truth table as defining the curl symbol:

p	$\sim p$
T	F
F	T

When two statements are combined disjunctively by inserting the word 'or' between them, the resulting compound statement is a *disjunction* (or *alternation*), and the two statements so combined are called *disjuncts* (or *alternatives*). The word 'or' has two different senses, one of which is clearly intended in the

statement 'Premiums will be waived in the event of sickness or unemployment'. The intention here is obviously that premiums are waived not only for sick persons and for unemployed persons, but also for persons who are both sick and unemployed. This sense of the word 'or' is called *weak* or *inclusive*. Where precision is at a premium, as in contracts and other legal documents, this sense is made explicit by use of the phrase 'and/or'.

A different sense of 'or' is intended when a restaurant lists 'tea or coffee' on its table d'hôte menu, meaning that for the stated price of the meal the customer can have one or the other, but *not both*. This second sense of 'or' is called *strong* or *exclusive*. Where precision is at a premium and the exclusive sense of 'or' is intended, the phrase 'but not both' is often added.

A disjunction which uses the inclusive 'or' asserts that *at least one disjunct is true*, whereas one which uses the exclusive 'or' asserts that *at least one disjunct is true but not both are true*. The *partial common meaning* that at least one disjunct is true, is the whole meaning of an inclusive disjunction and a part of the meaning of an exclusive disjunction.

In Latin, the word 'vel' expresses the inclusive sense of the word 'or' and the word 'aut' expresses the exclusive sense. It is customary to use the first letter of 'vel' to symbolize 'or' in its inclusive sense. Where p and q are any two statements whatever, their weak or inclusive disjunction is written $p \vee q$. The symbol ' $\vee$ ', called a *wedge* (or a *vee*), is a truth-functional connective, and is defined by the following truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

An obviously valid argument containing a disjunction is the following Disjunctive Syllogism:

The United Nations will become more responsible or there will be a third world war.

The United Nations will not become more responsible.

Therefore, there will be a third world war.

It is evident that a Disjunctive Syllogism is valid on *either* interpretation of the word 'or', that is, regardless of whether its first premiss asserts an inclusive or exclusive disjunction. It is usually difficult, and sometimes impossible, to discover which sense of the word 'or' is intended in a disjunction. But the typical valid argument that has a disjunction for a premiss is, like the Disjunctive Syllogism, valid on either interpretation of the word 'or'. Hence we effect a simplification by translating any occurrence of the word 'or' into the logical symbol ' $\vee$ '—*regardless of which sense of 'or' is intended*. Of course,

where it is explicitly stated that the disjunction is exclusive, by use of the added phrase 'but not both', for example, we do have the symbolic apparatus for expressing that sense, as will be explained below.

The use of parentheses, brackets, and braces for punctuating mathematical expressions is familiar. In the absence of a special convention, no number is uniquely denoted by the expression ' $6 + 9 \div 3$ ', although when punctuation makes clear how its constituents are to be grouped, it denotes either 5 or 9. Punctuation is needed to resolve ambiguity in the language of symbolic logic too, since compound statements may themselves be combined to produce more complicated compounds. Ambiguity is present in $p \cdot q \vee r$, which could be either the conjunction of p with $q \vee r$, or else the disjunction of $p \cdot q$ with r . These two different senses are unambiguously given by different punctuations: $p \cdot (q \vee r)$ and $(p \cdot q) \vee r$. In case p and q are both false and r is true, the first punctuated expression is false (since its first conjunct is false), but the second punctuated expression is true (since its second disjunct is true). Here a difference in punctuation makes all the difference between truth and falsehood. In symbolic logic, as in mathematics, we use parentheses, brackets, and braces for punctuation. To cut down on the number of punctuation marks required, however, we establish the symbolic convention that in any expression the curl will apply to the smallest component that the punctuation permits. Thus the ambiguity of $\sim p \vee q$, which might mean either $(\sim p) \vee q$ or $\sim(p \vee q)$, is resolved, by our convention, to mean the first of these. The curl can, and therefore by our convention *does*, apply to the first component p rather than to the larger expression $p \vee q$.

The word 'either' has a variety of different uses in English. It has conjunctive force in 'The Disjunctive Syllogism' is valid on either interpretation of the word 'or'. It frequently serves merely to introduce the first disjunct in a disjunction, as in 'Either the United Nations will become more responsible or there will be a third world war'. Perhaps the most useful function of the word 'either' is to punctuate some compound statements. Thus the sentence

More stringent antipollution measures will be enacted and the laws will be strictly enforced or the quality of life will be degraded still further.

can have its ambiguity resolved in one direction by placing the word 'either' at its beginning, or in the other direction by inserting the word 'either' right after the word 'and'. Such punctuation is effected in our symbolic language by parentheses. The ambiguous formula $p \cdot q \vee r$ discussed in the preceding paragraph corresponds to the ambiguous sentence considered in this one. The two different punctuations of the formula correspond to the two different punctuations of the sentence effected by the two different insertions of the word 'either'.

Not all conjunctions are formulated by explicitly placing the word 'and' between complete sentences, as in 'Charlie's neat and Charlie's sweet'. Indeed, the latter would more naturally be expressed as 'Charlie's neat and sweet'. The

familiar 'Jack and Jill went up the hill' is the more natural way of expressing the conjunction 'Jack went up the hill and Jill went up the hill'. It is the same with disjunctions: 'Either Alice or Betty will be elected' expresses more briefly the proposition alternatively formulated as 'Either Alice will be elected or Betty will be elected'; and 'Charlene will be either secretary or treasurer' expresses somewhat more briefly the same proposition as 'Either Charlene will be secretary or Charlene will be treasurer'.

The negation of a disjunction is often expressed by using the phrase 'neither-nor'. Thus the disjunction 'Either Alice or Betty will be elected' is denied by the statement 'Neither Alice nor Betty will be elected'. The disjunction would be symbolized as $A \vee B$ and its negation as either $\sim(A \vee B)$ or as $(\sim A) \cdot (\sim B)$. (The logical equivalence of these two formulas will be discussed in Section 2.4.) To deny that at least one of two statements is true is to assert that both of the two statements are false.

The word 'both' serves various functions. One is simply a matter of emphasis. To say 'Both Jack and Jill went up the hill' is only to emphasize that the two of them did what they are asserted to have done by saying 'Jack and Jill went up the hill'. A more useful function of the word 'both' is punctuational, like that of 'either'. 'Both . . . and --- are not ---' is used to make the same statement as 'neither . . . nor --- is ---'. In such sentences, the *order* of the words 'both' and 'not' is very significant. There is a great difference between

Alice and Betty will not both be elected.

and

Alice and Betty will both not be elected.

The former would be symbolized as $\sim(A \cdot B)$, the latter as $(\sim A) \cdot (\sim B)$.

Finally, it should be remarked that the word 'unless' can also be used in expressing the disjunction of two statements. Thus 'Our resources will soon be exhausted unless more recycling of materials is effected' and 'Unless more recycling of materials is effected our resources will soon be exhausted' can equally well be expressed as 'Either more recycling of materials is effected or our resources will soon be exhausted' and symbolized as $M \vee E$.

Since an exclusive disjunction asserts that at least one of its disjuncts is true but that they are not both true, we can symbolize the exclusive disjunction of any two statements p and q quite simply as $(p \vee q) \cdot \sim(p \cdot q)$. Thus we are able to symbolize conjunctions, negations, and both inclusive and exclusive disjunctions. Any compound statement that is built up out of simple statements by repeated use of truth-functional connectives will have its truth value completely determined by the truth values of those simple statements. For example, if A and B are true statements and X and Y are false, the truth value of the compound statement $\sim[(\sim A \vee X) \vee \sim(B \cdot Y)]$ can be discovered as follows. Since A is true, $\sim A$ is false, and since X is

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false also, the disjunction $(\sim A \vee X)$ is false. Since Y is false, the conjunction $(B \cdot Y)$ is false, and so its negation $\sim(B \cdot Y)$ is true. Hence the disjunction $(\sim A \vee X) \vee \sim(B \cdot Y)$ is true, and its negation, which is the given statement, is false. Such a stepwise procedure, beginning with the inmost components, always permits us to determine the truth value of a truth-functionally compound statement given the truth values of its component simple statements.

EXERCISES³

I. If A and B are true statements and X and Y are false statements, which of the following compound statements are true?

*1. $\sim(A \vee X)$

2. $\sim A \vee \sim X$

3. $\sim B \cdot \sim Y$

4. $\sim(B \cdot Y)$

11. $A \vee [X \cdot (B \vee Y)]$

12. $X \vee [A \cdot (Y \vee B)]$

13. $\sim\{\sim[\sim(A \cdot \sim X) \cdot \sim A] \cdot \sim X\}$

14. $\sim\{\sim[\sim(A \cdot \sim B) \cdot \sim A] \cdot \sim A\}$

2.2 Conditional Statements

The compound statement 'If the train is late, then we shall miss our connection' is a *conditional* (or a *hypothetical*, an *implication*, or an *implicative statement*). The component between the 'if' and the 'then' is called the *antecedent* (or the *implicans* or *protasis*), and the component that follows the 'then' is the *consequent* (or the *implicate* or *apodosis*). A conditional does not state either that its antecedent is true or that its consequent is true; it says only that if its antecedent is true, then its consequent is true also, that is, its antecedent *implies* its consequent. The key to the meaning of a conditional is the relation of *implication* that is asserted to hold between its antecedent and its consequent, in that order.

If we examine a number of different conditionals, we can see that there are different kinds of implications which they may express. In the conditional 'If all cats like liver and Dinah is a cat, then Dinah likes liver', the consequent follows *logically* from its antecedent. On the other hand, in the conditional 'If the figure is a triangle, then the figure has three sides', the consequent follows from the antecedent by the very *definition* of the word 'triangle'. The truth of the conditional 'If this piece of gold is placed in *aqua regia*, then this piece of gold dissolves' is not a matter of either logic or definition. The connection asserted here is *causal* and must be discovered empirically. These examples show that there are different kinds of implications that constitute different senses of the 'if-then' phrase. Having noted these differences, we now seek to find some identifiable common meaning, some partial meaning that is common to these different types of conditionals.

Our discussion of 'if-then' will parallel our previous discussion of the word 'or'. First, we pointed out two different senses of that word. Second, we noted that there was a common partial meaning: That *at least one disjunct is true* was seen to be involved in both the inclusive and the exclusive 'or'. Third, we introduced the special symbol ' $\vee$ ' to represent this common partial meaning (which was the whole meaning of 'or' in its inclusive sense). Fourth, we observed that because arguments like the Disjunctive Syllogism are valid on either interpretation of the word 'or', symbolizing *any* occurrence of the word 'or' by the wedge symbol, preserves the validity of such arguments. And because we are interested in arguments from the point of view of determining their validity, this translation of the word 'or' into ' $\vee$ ', which may abstract from or ignore part of its meaning in some cases, is wholly adequate for our present purposes.

A common partial meaning of these different kinds of conditional statements emerges when we ask what circumstances would suffice to establish the *falsehood* of a conditional. Under what circumstances would we agree that the conditional 'If this piece of gold is placed in this solution, then this piece of gold dissolves' is false? Clearly, the statement is false in the case that this piece of gold is actually placed in this solution and *does not dissolve*. Any conditional

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with a true antecedent and a false consequent must be false. Hence any conditional *if p, then q* is known to be false in case the conjunction $p \cdot \sim q$ is known to be true, that is, in case the antecedent of the conditional is true and its consequent false. For the conditional to be true, the indicated conjunction must be false, which means that the negation of that conjunction must be true. In other words, for any conditional *if p, then q* to be true, $\sim(p \cdot \sim q)$, the negation of the conjunction of its antecedent with the negation of its consequent, must be true also. We may, then, regard the latter as a *part* of the meaning of the former.

We introduce the new symbol ' $\supset$ ', called a *horseshoe*, to represent the partial meaning that is common to all conditional statements. We define ' $p \supset q$ ' as an abbreviation for ' $\sim(p \cdot \sim q)$ '. The horseshoe is a truth-functional connective, whose exact significance is indicated by the following truth table:

p	q	$\sim q$	$p \cdot \sim q$	$\sim(p \cdot \sim q)$	$p \supset q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T

Here, the first two columns represent all possible truth values for the component statements p and q , and the third, fourth, and fifth represent successive stages in determining the truth value of the compound statement $\sim(p \cdot \sim q)$ in each case. The sixth column is identically the same as the fifth, since the formulas that head them are defined to express the same proposition. The horseshoe symbol must not be thought of as representing *the* meaning of 'if-then', or *the* relation of implication, but rather symbolizes a common partial factor of the various different kinds of implications signified by the 'if-then' phrase.

We can regard the horseshoe as symbolizing a special, extremely weak kind of implication. It is expedient for us to do so, since convenient ways to read ' $p \supset q$ ' are 'if p , then q ', ' p implies q ', or ' p only if q '. The weak implication symbolized by ' $\supset$ ' is called a *material implication*. Its special name indicates that it is a special concept and not to be confused with the other, more usual kinds of implication. Some conditional statements in English do express merely material implications, as for example 'If Russia is a democracy, then I'm a Dutchman'. It is clear that the implication expressed here is neither logical, definitional, nor causal. No 'real connection' is alleged to hold between what the antecedent states and what is stated by the consequent. This sort of conditional is ordinarily intended as an emphatic or humorous method of denying the truth of its antecedent, for it typically contains a notoriously or ridiculously false statement as its consequent. Any such assertion about truth values is adequately symbolized by using the truth-functional connective ' $\supset$ '.

Although most conditional statements express more than a merely material

implication between antecedent and consequent, we now propose to symbolize *any* occurrence of 'if-then' by the truth-functional connective ' $\supset$ '. Such symbolizing abstracts from or ignores part of the meaning of most conditional statements. But the proposal can be justified on the grounds that the validity of valid arguments involving conditionals is preserved when the conditionals are regarded as expressing material implications only, as will be established in the following section.

Conditional statements can be expressed in a variety of ways. A statement of the form 'if p , then q ' could equally well be expressed as 'if p , q ', ' q if p ', 'that p implies that q ', 'that p entails that q ', ' p only if q ', 'that p is a sufficient condition that q ', or as 'that q is a necessary condition that p ', and any of these formulations will be symbolized as $p \supset q$.